THE SUBALGEBRAS OF A_2

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Abstract. A classification of the semisimple subalgebras of the Lie algebra of traceless 3×3 matrices with complex entries, denoted A_2, is well-known. We classify its nonsemisimple subalgebras, thus completing the classification of the subalgebras of A_2.

1. Introduction

The study of subalgebras of semisimple Lie algebras has largely focused on semsimple subalgebras. Notable examples include the work of Dynkin [5], and Minchenko [10], which combine to classify the simple subalgebras of the exceptional Lie algebras, up to inner automorphism. Another important example is de Graaf's classification of the semisimple subalgebras of the simple Lie algebras of ranks ≤ 8, up to linear equivalence, which is somewhat weaker than a classification up to inner automorphism [7, 8].

Less is known about nonsemisimple subalgebras of semisimple Lie algebras. Levi's Theorem [[9], Chapter III, Section 9] implies that

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subalgebras must be either semisimple, solvable, or Levi decomposable. A subalgebra is Levi decomposable if it is a semidirect or direct sum of a semisimple subalgebra and a solvable subalgebra.

We have made considerable progress towards classifying solvable and Levi decomposable subalgebras of semisimple Lie algebras. For instance, in [2], we classified the abelian extensions of the special orthogonal Lie algebras $\mathfrak{so}(2n,\mathbb{C})$ in the exceptional Lie algebras E_{n+1}, up to inner automorphism. In [1] we classified an important family of Levi decomposable subalgebras of the classical Lie algebras. Most recently, we classified all subalgebras of $A_1 \oplus A_1$, up to inner automorphism [4]. A_1 is the special Lie algebra of traceless 2×2 matrices with complex entries.

In the present article, we seek to extend the knowledge of subalgebras, especially nonsemisimple subalgebras, of semisimple Lie algebras. In particular, we classify the solvable, and Levi decomposable subalgebras of the simple Lie algebra A_2 of traceless 3×3 matrices with complex entries. Since its semisimple subalgebras are well-known [7, 8], Levi's Theorem implies that the classification of the subalgebras of A_2 will be complete. The classification is up to inner automorphism.

The article is organized as follows. Section 2 contains basic review of the simple Lie algebra A_2 and its semisimple subalgebras. We describe a classification of solvable Lie algebras for degrees ≤ 3 in Section 3 that will be used in a proceeding section. Additional notation and terminology that will be used in the article are recorded in Section 4. In Section 5 we classify the Levi decomposable subalgebras of A_2. Finally, in Section 6, we classify the solvable subalgebras of A_2.

2. The simple Lie algebra A_2

The special linear algebra A_2 is the Lie algebra of traceless 3×3 matrices with complex entries. A Chevalley basis $\{x_i, y_i, h_j : 1 \leq i \leq 3, 1 \leq j \leq 2\}$ of A_2 is defined as follows:

$$\begin{aligned} &ah_1 + bh_2 + cx_1 + dx_2 + ex_3 + c'y_1 + d'y_2 + e'y_3 \\ &= \begin{pmatrix} a & c & -e \\ c' & b-a & d \\ -e' & d' & -b \end{pmatrix}. \end{aligned} \tag{1}$$

where $a, b, c, d, e, c', d', e' \in \mathbb{C}$.

From [7, 8], there are precisely two semisimple subalgebras of A_2 up to linear equivalence, both isomorphic to the special linear algebra A_1 of traceless 2×2 matrices with complex entries. By [[10], Theorem

3.3], this is also true up to inner automorphism. They are

$$\begin{aligned} A_1^1 &\equiv \langle x_3, y_3, h_1 + h_2\rangle, \\ A_1^2 &\equiv \langle x_1 + x_2, 2y_1 + 2y_2, 2h_1 + 2h_2\rangle. \end{aligned} \tag{2}$$

3. Solvable Lie algebras of small dimension

A full classification of solvable Lie algebras is not known and thought to be an impossible task. However, classifications of solvable Lie algebras in special cases have been considered (e.g., [6, 11, 12, 13, 14, 15]). For instance, de Graaf classified the solvable Lie algebras in dimension ≤ 4 over fields of any characteristic [6].

With $\mathbb{C}$ as the ground field, we describe this classification in totality for dimension ≤ 3, and describe that portion of the dimension 4 classification which is relevant to the present article. In each case, $z_1, z_2, ..., z_n$ are basis elements of the n-dimensional, solvable Lie algebra being described, and all nonzero commutation relations for that algebra are presented.

$$J \quad \text{The abelian Lie algebra of dimension 1} \tag{3}$$

$$\begin{array}{ll} K_1 & \text{The abelian Lie algebra of dimension 2} \\ K_2 & [z_1, z_2] = z_1 \end{array} \tag{4}$$

$$\begin{array}{ll} L_1 & \text{The abelian Lie algebra of dimension 3} \\ L_2 & [z_3, z_1] = z_1, [z_3, z_2] = z_2 \\ L_{3,a} & [z_3, z_1] = z_2, [z_3, z_2] = az_1 + z_2 \\ L_4 & [z_3, z_1] = z_2, [z_3, z_2] = z_1 \\ L_5 & [z_3, z_1] = z_2 \end{array} \tag{5}$$

$$\begin{array}{ll} M_8 & [z_1, z_2] = z_2, [z_3, z_4] = z_4 \\ M_{12} & [z_4, z_1] = z_1, [z_4, z_2] = 2z_2, [z_4, z_3] = z_3, [z_3, z_1] = z_2 \\ M_{13,a} & [z_4, z_1] = z_1 + az_3, [z_4, z_2] = z_2, [z_4, z_3] = z_1, [z_3, z_1] = z_2 \\ M_{14} & [z_4, z_1] = z_3, [z_4, z_3] = z_1, [z_3, z_1] = z_2 \end{array} \tag{6}$$

Note that we get a nonisomorphic solvable Lie algebra $L_{3,a}$ for each $a \in \mathbb{C}$; and a nonisomorphic solvable Lie algebra $M_{13,a}$ for each $a \in \mathbb{C}$.

4. Additional definitions and notation

- Let $\mathfrak{g}$ be a Lie algebra, then $Aut(\mathfrak{g})$ is the group of automorphisms of $\mathfrak{g}$, $Inn(\mathfrak{g})$ is the group of inner automorphisms of $\mathfrak{g}$, and $Der(\mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$.
- Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras, and $\varphi : \mathfrak{g} \to Der(\mathfrak{g}')$ a Lie algebra homomorphism. Consider the Levi decomposable algebra $\mathfrak{g} \Subset_\varphi \mathfrak{g}'$ with $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$. Then, $[x, y] \equiv \varphi(x)(y)$.

- Let $W_1, ..., W_m \in A_2$. Then,
$$\langle W_1, ..., W_m \rangle$$
is the subalgebra of A_2 generated by W_1,...,W_m. In this article, W_1,...,W_m will generally not be a minimal generating set.
- Let φ and ϱ be Lie algebra embeddings of $\mathfrak{g}'$ into $\mathfrak{g}$. If φ and ϱ are equivalent up to inner automorphism we write $\varphi \sim \varrho$.
- Two embeddings φ and ϱ of $\mathfrak{g}'$ into $\mathfrak{g}$ are *linearly equivalent* if for each representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced $\mathfrak{g}'$-representations $\pi \circ \varphi$, $\pi \circ \varrho$ are equivalent, and we write $\varphi \sim_L \varrho$.

We define linear equivalence of subalgebras much as we did for embeddings.

- Two subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ of $\mathfrak{g}$ are linearly equivalent if for every representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the subalgebras $\rho(\mathfrak{g}')$, $\rho(\mathfrak{g}'')$ of $\mathfrak{gl}(V)$ are conjugate under $GL(V)$. Clearly if two embeddings or subalgebras are equivalent up to inner automorphism, then they are linearly equivalent. But, the converse is not in general true.

5. The Levi decomposable subalgebras of A_2

To determine the Levi decomposable subalgebras of A_2, we first decompose A_2 with respect to the adjoint actions of A_1^1 and A_1^2, respectively:

$$\begin{array}{rlccccccc} A_2 & \cong_{A_1^1} & \langle x_3, y_3, h_1 + h_2 \rangle & \oplus & \langle x_2, y_1 \rangle & \oplus & \langle x_1, y_2 \rangle & \oplus & \\ & & \langle h_1 - h_2 \rangle & & & & & & \\ & \cong_{A_1^1} & V(2) & \oplus & V(1) & \oplus & V(1) & \oplus & V(0), \end{array} \tag{7}$$

$$\begin{array}{rlccc} A_2 & \cong_{A_1^2} & \langle x_1 + x_2, 2y_1 + 2y_2, 2h_1 + 2h_2 \rangle & \oplus & \\ & & \langle x_3, x_1 - x_2, h_1 - h_2, y_1 - y_2, y_3 \rangle & & \\ & \cong_{A_1^2} & V(2) & \oplus & V(4), \end{array} \tag{8}$$

where $V(n)$ is the $n + 1$ dimensional, irreducible representation of A_1. In each decomposition $V(2) \cong A_1$ as a subalgebra of A_2. The remaining representations in the decompositions, or combinations thereof, give us the potential solvable components for Levi decomposable subalgebras.

Lemma 5.1. *Let $\psi : \mathfrak{g} \Subset \mathfrak{s} \to \mathfrak{h} \Subset \mathfrak{r}$ be a Lie algebra isomorphism, where $\mathfrak{g}$ and $\mathfrak{h}$ are semisimple and $\mathfrak{s}$ and $\mathfrak{r}$ are solvable. Then $\psi(\mathfrak{s}) = \mathfrak{r}$.*

Proof. Let $\pi : \mathfrak{h} \Subset \mathfrak{r} \to \mathfrak{h}$ be the projection map of $\mathfrak{h} \Subset \mathfrak{r}$ onto $\mathfrak{h}$. Then, $\pi(\psi(\mathfrak{s}))$ is a solvable ideal of $\mathfrak{h}$. Since $\mathfrak{h}$ is semisimple, $\pi(\psi(\mathfrak{s})) = 0$. Hence, $\psi(\mathfrak{s}) \subseteq \mathfrak{r}$. The Levi factor in a Levi decomposition is unique, up

to isomorphism (i.e., $\mathfrak{g} \cong \mathfrak{h}$). Hence, dimension considerations imply $\psi(\mathfrak{s}) = \mathfrak{r}$. $\square$

Theorem 5.2. *A classification of Levi decomposable subalgebras of A_2, up to inner automorphism, is given by:*

$$
\begin{array}{lcl}
\langle x_3, y_3, h_1+h_2\rangle \oplus \langle h_1-h_2\rangle & \cong & A_1 \oplus J, \\
\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle & \cong & A_1 \Subset_{\varphi_1} K_1, \\
\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle & \cong & A_1 \Subset_{\varphi_1} K_1, \\
\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle & \cong & A_1 \Subset_{\varphi_2} L_2, \\
\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle & \cong & A_1 \Subset_{\varphi_2} L_2,
\end{array} \tag{9}
$$

where the Lie algebra homomorphisms $\varphi_1 : A_1 \to Der(K_1)$; and $\varphi_2 : A_1 \to Der(L_2)$ are defined in the proof below.

Proof. The decomposition of A_2 with respect to the adjoint action of A_1^2 in Eq. (8) contains exactly two components: $\langle x_1+x_2, 2y_1+2y_2, 2h_1+2h_2\rangle \cong A_1$ and $\langle x_3, x_1-x_2, h_1-h_2, y_1-y_2, y_3\rangle$. Thus, the only nontrivial extension of A_1^2 in A_2 is A_2 itself. This implies that A_1^2 cannot be extended to a Levi decomposable subalgebra of A_2. We now turn our attention to A_1^1.

In the decomposition of A_2 with respect to the adjoint action of A_1^1 in Eq. (7), the component $\langle x_3, y_3, h_1+h_2\rangle$ is isomorphic to A_1, and the other components, or combinations thereof, give us the potential extensions of $\langle x_3, y_3, h_1+h_2\rangle \cong A_1$.

The component $\langle h_1-h_2\rangle$ is one-dimensional, and is therefore a subalgebra of A_2. Since it is the only one-dimensional component, the only one-dimensional extension of $\langle x_3, y_3, h_1+h_2\rangle$, up to inner automorphism, is

$$\langle x_3, y_3, h_1+h_2\rangle \oplus \langle h_1-h_2\rangle. \tag{10}$$

Note that the sum above is direct. We have

$$A_1 \oplus J \;\cong\; \langle x_3, y_3, h_1+h_2\rangle \oplus \langle h_1-h_2\rangle. \tag{11}$$

We now consider 2-dimensional extensions of A_1^1 in A_2. The possible 2-dimensional extensions of A_1^1 are by an irreducible A_1-representation $V(1)$ with respect to the adjoint action of A_1^1. From Eq. (7), a highest weight vector of such a representation is a nonzero linear combination of x_1 and x_2. However, only (a nonzero multiple of) x_1 or x_2 generates a representation isomorphic to $V(1)$ with respect to the adjoint action of A_1^1 which is also a 2-dimensional subalgebra of A_2.

Hence, the possible 2-dimensional extensions of A_1^1 are by $\langle x_1, y_2\rangle$, or $\langle x_2, y_1\rangle$. Note that both $\langle x_1, y_2\rangle$, and $\langle x_2, y_1\rangle$ are abelian subalgebras,

and hence isomorphic to K_1. We thus have (at most) two 5-dimensional Levi decomposable subalgebras of A_2:

$$\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle, \quad \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle. \tag{12}$$

These subalgebras are isomorphic. The Chevalley involution of A_2 induces an isomorphism between $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle$ and $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle$.

Define

$$\begin{array}{rccc} \chi: & A_1 & \hookrightarrow & A_2 \\ & x & \mapsto & x_3 \\ & y & \mapsto & y_3 \\ & h & \mapsto & h_1+h_2, \end{array} \tag{13}$$

where $\{x, y, h\}$ is a Chevalley basis of A_1, and

$$\begin{array}{rcccl} \psi_1: & K_1 & \to & K_1 & \subseteq A_2, \\ & z_1 & \mapsto & x_1 & \\ & z_2 & \mapsto & y_2 & \end{array} \tag{14}$$

$$\begin{array}{rccl} \varphi_1: & A_1 & \to & Der(K_1) \\ \varphi_1(L): & z_i & \mapsto & \psi_1^{-1}([\chi(L), \psi_1(z_i)]), i = 1, 2, \end{array} \tag{15}$$

then

$$\begin{array}{rcl} A_1 \Subset_{\varphi_1} K_1 & \cong & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle, \\ A_1 \Subset_{\varphi_1} K_1 & \cong & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle. \end{array} \tag{16}$$

We now show that $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle$ and $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle$ are inequivalent subalgebras of A_2. By way of contradiction, suppose that they are equivalent. Then, there exists $A \in SL(3, \mathbb{C})$ such that $A^{-1}\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle A = \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle$.

By Lemma 5.1, $A^{-1}\langle x_1, y_2\rangle A = \langle x_2, y_1\rangle$. However, no such $A \in SL(3, \mathbb{C})$ exists, as one can show by direct calculation. Hence, $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle$ and $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle$ are inequivalent subalgebras of A_2.

We now consider 3-dimensional extensions of A_1^1 in A_2. By Eq. (7), a 3-dimensional extension of A_1^1 with respect to the adjoint action of A_1^1 must be isomorphic to $V(1) \oplus V(0)$. The highest weight vector of $V(0)$ must be (a nonzero scalar multiple of) $h_1 - h_2$, and the highest weight vector of $V(1)$ must be a nonzero linear combination of x_1 and x_2. As above, it is only a scalar multiple of x_1 or x_2 that can contribute to a representation of $V(1) \oplus V(0)$ which is also a 3-dimensional subalgebra.

Hence, the possible 3-dimensional extensions of A_1^1 are $\langle h_1 - h_2, x_1, y_2\rangle$, and $\langle h_1 - h_2, x_2, y_1\rangle$, both of which are isomorphic to $L_{3,0}$. It follows

that we have (at most) two 6-dimensional Levi decomposable subalgebras of A_2:

$$\begin{aligned}\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle,\\ \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle.\end{aligned} \tag{17}$$

These subalgebras are isomorphic. The Chevalley involution of A_2 induces an isomorphism between $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle$ and $\langle x_3, y_3, h_1-h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle$.

Define

$$\begin{array}{rcccl} \psi_2: & L_2 & \to & L_2 & \subseteq A_2,\\ & z_1 & \mapsto & x_1 & \\ & z_2 & \mapsto & y_2 & \\ & z_3 & \mapsto & \frac{1}{3}(h_1-h_2), & \end{array} \tag{18}$$

$$\begin{array}{rccl} \varphi_2: & A_1 & \to & Der(L_2)\\ \varphi_2(L): & z_i & \mapsto & \psi_2^{-1}([\chi(L), \psi_2(z_i)]), i=1,2,3, \end{array} \tag{19}$$

then

$$\begin{array}{rcl} A_1 \Subset_{\varphi_2} L_2 & \cong & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle,\\ A_1 \Subset_{\varphi_2} L_2 & \cong & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle. \end{array} \tag{20}$$

We now show that $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle$ and $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle$ are inequivalent subalgebras of A_2. By way of contradiction, suppose that they are equivalent. Then, there exists $A \in SL(3,\mathbb{C})$ such that $A^{-1}\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle A = \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle$.

By Lemma 5.1, $A^{-1}\langle h_1-h_2, x_1, y_2\rangle A = \langle h_1-h_2, x_2, y_1\rangle$. However, no such $A \in SL(3,\mathbb{C})$ exists, as one can show by direct calculation. Hence, $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle$ and $\langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle$ are inequivalent subalgebras of A_2.

Finally we consider 4-dimensional extensions of A_1^1 in A_2. From Eq. (7), such an extension must be isomorphic to $V(1) \oplus V(1)$ as a representation with respect to the adjoint action of A_1^1. However, as a subalgebra of A_2, such a representation will generate a subalgebra of dimension greater than 4. Thus there are no 4-dimensional extensions of A_1^1 in A_2. □

If we make the following definitions:

$$\begin{array}{lcl} (A_1 \oplus J)^1 & \equiv & \langle x_3, y_3, h_1+h_2\rangle \oplus \langle h_1-h_2\rangle,\\ (A_1 \Subset_{\varphi_1} K_1)^1 & \equiv & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_1, y_2\rangle,\\ (A_1 \Subset_{\varphi_1} K_1)^2 & \equiv & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle x_2, y_1\rangle,\\ (A_1 \Subset_{\varphi_2} L_2)^1 & \equiv & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_1, y_2\rangle,\\ (A_1 \Subset_{\varphi_2} L_2)^2 & \equiv & \langle x_3, y_3, h_1+h_2\rangle \Subset \langle h_1-h_2, x_2, y_1\rangle, \end{array} \tag{21}$$

we get the concise summary of Levi decomposable subalgebras of A_2 in Table 1 at the end of the paper.

Remark 5.3. Note that $\mathfrak{gl}(2,\mathbb{C}) \cong A_1 \oplus J$, where $\mathfrak{gl}(2,\mathbb{C})$ is the general linear algebra of 2×2 complex matrices. By Theorem 5.2, there is a unique subalgebra of A_2 isomorphic to $\mathfrak{gl}(2,\mathbb{C})$.

6. The solvable subalgebras of A_2

6.1. **Introduction.** We shall proceed by cases to classify the solvable subalgebras of A_2 for each isomorphism class.

6.2. One-dimensional subalgebras of A_2.

Theorem 6.1. *A classification of* 1*-dimensional (solvable) subalgebras of* A_2*, up to inner automorphism, is given by:*

$$\begin{aligned} J^1 &= \langle x_1 + x_2 \rangle, \\ J^2 &= \langle x_1 \rangle, \\ J^3 &= \langle h_1 + 2h_2 + x_1 \rangle, \\ J^{4,\alpha} &= \langle h_1 + \alpha h_2 \rangle, \end{aligned} \tag{22}$$

for $\alpha \in \mathbb{C}$*, where* $J^{4,\alpha} \sim J^{4,\beta}$ *if and only if* $\alpha = 1-\beta$, $\alpha = \beta$, $\alpha = (-1+\alpha)\beta$, $\alpha = -(-1+\alpha)(-1+\beta)$, $\alpha\beta = 1$, *or* $-\alpha(-1+\beta) = 1$. *Equivalently,* $J^{4,\alpha} \sim J^{4,\beta}$ *if and only if* $\beta = \alpha, \frac{1}{\alpha}, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{-1+\alpha}$*, or* $\frac{-1+\alpha}{\alpha}$.

Proof. Every element of A_2 is equivalent to a traceless matrix in Jordan Normal Form, via conjugation by an element of $GL(3,\mathbb{C})$, and hence by an element of $SL(3,\mathbb{C})$. Thus, every element of A_2 is equivalent to a matrix in one of the following forms:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}, \quad \text{or} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha-\beta \end{pmatrix}, \tag{23}$$

for some $\alpha, \beta \in \mathbb{C}$. Because multiplying by a nonzero scalar does not affect the block type of the Jordan Normal Form, a 1-dimensional subalgebra generated by one of the above matrices is inequivalent to any algebra generated by a matrix of either of the other two types.

Corresponding to the first matrix in Eq. (23), define the subalgebra

$$J^1 \equiv \langle x_1 + x_2 \rangle. \tag{24}$$

Any subalgebra generated by a matrix whose Jordan Canonical Form is the first matrix in Eq. (23) is equivalent to J^1.

The subalgebra generated by a matrix whose Jordan Canonical Form is the second matrix in Eq. (23) is equivalent to the subalgebra generated by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \begin{pmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{25}$$

for $\alpha \in \mathbb{C}^*$. The second matrix in Eq. (25) is conjugate in $SL(3, \mathbb{C})$ to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{26}$$

Hence, we have the following inequivalent 1-dimensional subalgbras

$$J^2 \equiv \langle x_1 \rangle, \quad J^3 \equiv \langle h_1 + 2h_2 + x_1 \rangle. \tag{27}$$

Finally, we consider 1-dimensional subalgebras generated by the third matrix in Eq. (23), which is diagonal. The 1-dimensional subalgebra generated by this matrix is equivalent to the subalgebra generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1+\alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix} = h_1 + \alpha h_2, \tag{28}$$

for some $\alpha \in \mathbb{C}$ (not necessarily the same α as in Eq. (23)). Define

$$J^{4,\alpha} \equiv \langle h_1 + \alpha h_2 \rangle. \tag{29}$$

Suppose $J^{4,\alpha} \sim J^{4,\beta}$. Then $h_1 + \alpha h_2$ is conjugate by an element of $SL(3, \mathbb{C})$ to $\lambda(h_1 + \beta h_2)$, for some $\lambda \in \mathbb{C}^*$. This implies that the diagonal matrix $h_1 + \alpha h_2$ must be equal to the diagonal matrix $\lambda(h_1 + \beta h_2)$ after permutation of diagonal entries. That is, $\{1, -1+\alpha, -\alpha\} = \{\lambda, \lambda(-1+\beta), -\lambda\beta\}$. We now proceed in cases according to the value of λ.

<u>Case 1</u>. $\lambda = 1$. Then $-\alpha = -1+\beta$, or $-\alpha = -\beta$. That is, $\alpha = 1-\beta$, or $\alpha = \beta$.

<u>Case 2</u>. $\lambda = -1+\alpha$. Then $-\alpha = -(-1+\alpha)\beta$, or $-\alpha = (-1+\alpha)(-1+\beta)$. That is, $\alpha = (-1+\alpha)\beta$, or $\alpha = -(-1+\alpha)(-1+\beta)$.

<u>Case 3</u>. $\lambda = -\alpha$. Then $1 = \alpha\beta$, or $-1+\alpha = \alpha\beta$. That is, $\alpha\beta = 1$, or $-\alpha(-1+\beta) = 1$.

We have shown that if $J^{4,\alpha} \sim J^{4,\beta}$, then $\alpha = 1-\beta$, $\alpha = \beta$, $\alpha = (-1+\alpha)\beta$, $\alpha = -(-1+\alpha)(-1+\beta)$, $\alpha\beta = 1$, or $-\alpha(-1+\beta) = 1$. Further, if one of these conditions holds then $J^{4,\alpha} \sim J^{4,\beta}$. $\square$

6.3. **Two-dimensional subalgebras of A_2.** We consider each of the two 2-dimensional (solvable) Lie algebras as subalgebras of A_2 separately in the theorems below.

Theorem 6.2. *A classification of the subalgebras of A_2 isomorphic to K_1, up to inner automorphism, is given by:*

$$\begin{aligned} K_1^1 &= \langle x_1 + x_2, x_3 \rangle, \\ K_1^2 &= \langle x_1, h_1 + 2h_2 \rangle, \\ K_1^3 &= \langle x_1, x_3 \rangle, \\ K_1^4 &= \langle x_1, y_2 \rangle, \\ K_1^5 &= \langle h_1, h_2 \rangle. \end{aligned} \tag{30}$$

Proof. Let K be a subalgebra which is isomorphic to K_1 with basis z_1, z_2 so that $[z_1, z_2] = 0$. From Theorem 6.1, we have the following cases.

<u>Case 1</u>. $\langle z_1 \rangle$ is conjugate in $SL(3,\mathbb{C})$ to $\langle x_1 + x_2 \rangle$. After conjugation, we may assume $z_1 = x_1 + x_2$. The commutation relation $[z_1, z_2] = 0$ implies $z_2 = a(x_1 + x_2) + bx_3$, for $a, b \in \mathbb{C}$. Hence, K is equivalent to $\langle x_1 + x_2, x_3 \rangle$. Define

$$K_1^1 \equiv \langle x_1 + x_2, x_3 \rangle. \tag{31}$$

<u>Case 2</u>. $\langle z_1 \rangle$ is conjugate in $SL(3,\mathbb{C})$ to $\langle x_1 \rangle$. After conjugation by an element of $SL(3,\mathbb{C})$, we may assume $z_1 = x_1$. The commutation relation $[z_1, z_2] = 0$ implies $z_2 = \alpha(h_1 + 2h_2) + \beta x_1 + \gamma x_3 + \delta y_2$, for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. This implies K is equivalent to $\langle x_1, \alpha(h_1 + 2h_2) + \gamma x_3 + \delta y_2 \rangle$.

<u>Case 2.1</u>. $\alpha \neq 0$. Then $z_1 = x_1$, and $z_2 = \alpha(h_1 + 2h_2) + \gamma x_3 + \delta y_2$. Conjugation by

$$\begin{pmatrix} 1 & 0 & \frac{\gamma}{3\alpha} \\ 0 & 1 & 0 \\ 0 & \frac{\delta}{3\alpha} & 1 \end{pmatrix} \in SL(3,\mathbb{C}), \tag{32}$$

fixes $z_1 = x_1$ and

$$A^{-1} z_2 A = \begin{pmatrix} \alpha & -\frac{\gamma\delta}{3\alpha} & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}. \tag{33}$$

Hence, K is equivalent to $\langle x_1, h_1 + 2h_2 \rangle$. Define

$$K_1^2 \equiv \langle x_1, h_1 + 2h_2 \rangle. \tag{34}$$

Case 2.2. $\alpha = 0$, $\gamma \neq 0$, $\delta \neq 0$. Then $z_1 = x_1$, and $z_2 = \gamma x_3 + \delta y_2$. It follows that $K = \langle x_1, x_3 + \frac{\delta}{\gamma} y_2 \rangle$. If

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \in SL(3, \mathbb{C}), \text{ with } \lambda^3 = \frac{\delta}{\gamma}, \tag{35}$$

then $A^{-1} \langle x_1, x_3 + \frac{\delta}{\gamma} y_2 \rangle A = \langle x_1, x_3 + y_2 \rangle$. Further, $B^{-1} \langle x_1, x_3 + y_2 \rangle B = K_1^1$, where

$$B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in SL(3, \mathbb{C}). \tag{36}$$

Case 2.3. $\alpha = 0$, $\gamma \neq 0$, $\delta = 0$. Then $z_1 = x_1$, and $z_2 = \gamma x_3$, and K is equivalent to $\langle x_1, x_3 \rangle$. Define

$$K_1^3 \equiv \langle x_1, x_3 \rangle. \tag{37}$$

Case 2.4. $\alpha = 0$, $\gamma = 0$, $\delta \neq 0$. Then $z_1 = x_1$, and $z_2 = \delta y_2$, and K is equivalent to $\langle x_1, y_2 \rangle$. Define

$$K_1^4 \equiv \langle x_1, y_2 \rangle. \tag{38}$$

Case 3. $\langle z_1 \rangle$ is conjugate in $SL(3, \mathbb{C})$ to $\langle h_1 + 2h_2 + x_1 \rangle$, so we may assume $z_1 = h_1 + 2h_2 + x_1$. The commutation relation $[z_1, z_2] = 0$ implies $z_2 = a(h_1 + 2h_2) + bx_1$, for $a, b \in \mathbb{C}$. This implies that K is equivalent to $\langle x_1, h_1 + 2h_2 \rangle = K_1^2$.

Case 4. $\langle z_1 \rangle$ is conjugate in $SL(3, \mathbb{C})$ to $\langle h_1 + \alpha h_2 \rangle$, so we may assume $z_1 = h_1 + \alpha h_2$.

Case 4.1. $\alpha = 2$. The commutation relation $[z_1, z_2] = 0$ implies that

$$z_2 = \begin{pmatrix} \beta & \delta & 0 \\ \epsilon & -\beta + \gamma & 0 \\ 0 & 0 & -\gamma \end{pmatrix}, \tag{39}$$

for $\beta, \gamma, \delta, \epsilon \in \mathbb{C}$. Let $A \in SL(3, \mathbb{C})$ such that $A^{-1} z_1 A = z_1$, then

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \frac{1}{ad-bc} \end{pmatrix}, \tag{40}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$ is chosen so that it conjugates the upper left 2×2 block of z_2 to its Jordan

Normal Form, then A in Equation (40) fixes z_1 and conjugates z_2 to

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda-\mu \end{pmatrix}. \tag{41}$$

In the former case, K is equivalent to $\langle x_1, h_1+2h_2\rangle = K_1^2$. In the latter case, K is equivalent to $\langle h_1, h_2\rangle$. Define

$$K_1^5 \equiv \langle h_1, h_2\rangle. \tag{42}$$

<u>Case 4.2</u>. $\alpha = -1$. The commutation relation $[z_1, z_2] = 0$ implies that

$$z_2 = \begin{pmatrix} \beta & 0 & -\delta \\ 0 & -\beta+\gamma & 0 \\ -\epsilon & 0 & -\gamma \end{pmatrix}, \tag{43}$$

for $\beta, \gamma, \delta, \epsilon \in \mathbb{C}$. Let $A \in SL(2,\mathbb{C})$ such that $A^{-1}z_1A = z_1$, then

$$A = \begin{pmatrix} a & 0 & b \\ 0 & \frac{1}{ad-bc} & 0 \\ c & 0 & d \end{pmatrix}, \tag{44}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$ is chosen so that it conjugates the 2×2 matrix consisting of the outer corners of z_2 to its Jordan Normal Form, then A in Equation (44) fixes z_1 and conjugates z_2 to

$$\begin{pmatrix} \lambda & 0 & 1 \\ 0 & -2\lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda-\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}. \tag{45}$$

In the latter case, K is equivalent to $\langle h_1, h_2\rangle = K_1^5$. In the former case, K is equivalent to $\langle x_3, h_1 - h_2\rangle$, which is equivalent to K_1^2 via conjugation by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in SL(3,\mathbb{C}). \tag{46}$$

<u>Case 4.3</u>. $\alpha = \frac{1}{2}$. The commutation relation $[z_1, z_2] = 0$ implies that

$$z_2 = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & \beta & \delta \\ 0 & \epsilon & -\beta+\gamma \end{pmatrix}, \tag{47}$$

for $\beta, \gamma, \delta, \epsilon \in \mathbb{C}$. Let $A \in SL(2, \mathbb{C})$ such that $A^{-1}z_1A = z_1$, then

$$A = \begin{pmatrix} \frac{1}{ad-bc} & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \tag{48}$$

for $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. For appropriate choices of a, b, c, d the matrix A fixes z_1 and conjugates z_2 to

$$\begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} -\lambda-\mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}. \tag{49}$$

In the latter case, K is equivalent to $\langle h_1, h_2 \rangle = K_1^5$. In the former case, K is equivalent to $\langle x_2, h_1 + \frac{1}{2}h_2 \rangle$. Then $B^{-1}\langle x_2, h_1 + \frac{1}{2}h_2 \rangle B = \langle x_1, h_1 + 2h_2 \rangle = K_1^2$, where

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in SL(3, \mathbb{C}). \tag{50}$$

Case 4.4. $\alpha \neq 2, \frac{1}{2}, -1$. The commutation relation $[z_1, z_2] = 0$ implies that z_2 is diagonal. Hence K is equivalent to $\langle h_1, h_2 \rangle = K_1^5$.

The above cases establish that each subalgebra of A_2 isomorphic to K_1 is equivalent to $K_1^1, K_1^2, K_1^3, K_1^4$, or K_1^5. Direct linear algebraic methods establish that these five subalgebras are pairwise inequivalent. For instance, K_1^5 is not equivalent to any of the other subalgebras since an equivalence would imply x_1 or $x_1 + x_2$ is diagonalizable, which of course is not true. □

Theorem 6.3. *A classification of the subalgebras of A_2 isomorphic to K_2, up to inner automorphism, is given by:*

$$\begin{aligned} K_2^1 &= \langle x_1 + x_2, h_1 + h_2 \rangle, \\ K_2^2 &= \langle x_1, -\tfrac{1}{3}h_2 + \tfrac{1}{3}h_2 + x_3 \rangle, \\ K_2^3 &= \langle x_1, -\tfrac{2}{3}h_1 - \tfrac{1}{3}h_2 + y_2 \rangle, \\ K_2^{4,\alpha} &= \langle x_1, \alpha h_1 + (2\alpha + 1)h_2 \rangle, \end{aligned} \tag{51}$$

where $K_2^{4,\alpha} \sim K_2^{4,\beta}$ if and only if $\alpha = \beta$.

Proof. Let K be a subalgebra of A_2 isomorphic to K_2 and let $z_1, z_2 \in K$ be nonzero such that $[z_1, z_2] = z_1$. After conjugation in $SL(3, \mathbb{C})$, we may assume $z_1 \in J^1$, J^2, J^3, or $J^{4,\alpha}$, for some $\alpha \in \mathbb{C}$.
Case 1. $z_1 \in J^1 = \langle x_1 + x_2 \rangle$. After scalar multiplication, we may assume $z_1 = x_1 + x_2$. The commutation relation $[z_1, z_2] = z_1$ implies

$z_2 = -h_1 - h_2 + ax_3 + b(x_1 + x_2)$, for some $a, b \in \mathbb{C}$. Hence, $K = \langle x_1 + x_2, -h_1 - h_2 + ax_3\rangle$. After conjugation by

$$\begin{pmatrix} 1 & 0 & -\frac{a}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{C}), \tag{52}$$

we have $K \sim \langle x_1 + x_2, h_1 + h_2\rangle \equiv K_2^1$.
<u>Case 2</u>. $z_1 \in J^2 = \langle x_1\rangle$. After scalar multiplication, we may assume $z_1 = x_1$. The commutation relation $[z_1, z_2] = z_1$ implies $z_2 = ah_1 + (2a + 1)h_2 + bx_1 + cx_3 + dy_2$, for some $a, b, c, d \in \mathbb{C}$. This implies $K = \langle x_1, ah_1 + (2a + 1)h_2 + cx_3 + dy_2\rangle$.
<u>Case 2.1</u>. $a \neq -\frac{1}{3}, -\frac{2}{3}$. Conjugation by

$$\begin{pmatrix} 1 & -\frac{cd}{3a+2} & \frac{c}{3a+1} \\ 0 & 1 & 0 \\ 0 & \frac{d}{3a+2} & 1 \end{pmatrix} \in SL(3, \mathbb{C}) \tag{53}$$

yields $\langle x_1, ah_1 + (2a + 1)h_2\rangle$.
<u>Case 2.2.1</u>. $a = -\frac{1}{3}$, $c \neq 0$. Conjugation by

$$\begin{pmatrix} \lambda & -cd\lambda & 0 \\ 0 & \lambda & 0 \\ 0 & d\lambda & \frac{1}{\lambda^2} \end{pmatrix} \in SL(3, \mathbb{C}), \tag{54}$$

for $\lambda \in \mathbb{C}$ such that $\lambda^3 = c$, yields $\langle x_1, -\frac{1}{3}h_1 + \frac{1}{3}h_2 + x_3\rangle$.
<u>Case 2.2.2</u>. $a = -\frac{1}{3}$, $c = 0$. Conjugation by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{pmatrix} \in SL(3, \mathbb{C}) \tag{55}$$

yields $\langle x_1, -\frac{1}{3}h_1 + \frac{1}{3}h_2\rangle = \langle x_1, h_1 - h_2\rangle$.
<u>Case 2.3.1</u>. $a = -\frac{2}{3}$, $d \neq 0$. Conjugation by

$$\begin{pmatrix} \lambda & -\frac{c(\lambda^2-1)}{\lambda^2} & -c + \frac{c(e^2-1)}{e^2} \\ 0 & \lambda & 0 \\ 0 & 1 & \frac{1}{\lambda^2} \end{pmatrix} \in SL(3, \mathbb{C}), \tag{56}$$

where $\lambda \in \mathbb{C}$ such that $d\lambda^3 = 1$, yields $\langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2\rangle$.
<u>Case 2.3.2</u>. $a = -\frac{2}{3}$, $d = 0$. Conjugation by

$$\begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{C}) \tag{57}$$

yields $\langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2\rangle = \langle x_1, 2h_1 + h_2\rangle$.

Case 3. $z_1 \in J^3 = \langle h_1 + 2h_2 + x_1 \rangle$. After scalar multiplication, we may assume $z_1 = h_1 + 2h_2 + x_1$. The commutation relation $[z_1, z_2] = z_1$ has no solution, implying that this case does not yield a subalgebra isomorphic to K_2.

Case 4. $z_1 \in J^{4,\alpha} = \langle h_1 + \alpha h_2 \rangle$. After scalar multiplication, we may assume $z_1 = h_1 + \alpha h_2$. The commutation relation $[z_1, z_2] = z_1$ has no solution, implying that this case does not yield a subalgebra isomorphic to K_2.

By cases 1-4, each subalgebra of A_2 isomorphic to K_2 is equivalent to one of the following subalgebras:

$$\begin{array}{lcl} K_2^1 & \equiv & \langle x_1 + x_2, h_1 + h_2 \rangle, \\ K_2^2 & \equiv & \langle x_1, -\frac{1}{3}h_1 + \frac{1}{3}h_2 + x_3 \rangle, \\ K_2^3 & \equiv & \langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2 \rangle, \\ K_2^{4,\alpha} & \equiv & \langle x_1, \alpha h_1 + (2\alpha + 1)h_2 \rangle, \end{array} \tag{58}$$

for some $\alpha \in \mathbb{C}$.

If K_2^1 were equivalent to K_2^2, then there would exist $A \in SL(3, \mathbb{C})$ such that $A^{-1}K_2^1 A = K_2^2$. However, no such A exists, as can be check by direct computation. Similarly, K_2^3 and $K_2^{4,\alpha}$ are inequivalent to K_2^1; K_2^2 and K_2^3 are inequivalent; K_2^2 and $K^{4,\alpha}$ are inequivalent; and K_2^3 and $K^{4,\alpha}$ are inequivalent. Direct computation establishes that $K_2^{4,\alpha}$ is conjugate to $K_2^{4,\beta}$ if and only if $\alpha = \beta$. □

6.4. **Three-dimensional, solvable subalgebras of A_2.** In this subsection we use the following lemma from [6] (Lemma 2.1):

Lemma 6.4. *Let T be a solvable Lie algebra. Then there is a subalgebra $S \subset T$ of codimension 1, and $z \in T$ such that $T = \langle z \rangle \Subset S$.*

Theorem 6.5. *A classification of the 3-dimensional solvable subalgebras of A_2, up to inner automorphism, is given by:*

$$\begin{array}{lcl} L_2^1 & = & \langle x_1, x_3, 2h_1 + h_2\rangle, \\ L_2^2 & = & \langle x_1, y_2, h_1 - h_2\rangle, \\ L^1_{3,-\frac{1}{4}} & = & \langle x_1, x_3, 2h_1 + h_2 + x_2\rangle, \\ L^2_{3,-\frac{1}{4}} & = & \langle y_1, y_3, 2h_1 + h_2 + x_2\rangle, \\ L^1_{3,-\frac{2}{9}} & = & \langle x_1 + x_2, x_3, h_1 + h_2\rangle, \\ L^1_{3,0} & = & \langle x_1, h_1, h_2\rangle, \\ L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} & = & \langle x_1, x_3, (\alpha-1)h_1 + \alpha h_2\rangle, \\ L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} & = & \langle x_1, y_2, h_1 + \alpha h_2\rangle, \\ L_4^1 & = & \langle x_1, x_3, h_2\rangle, \\ L_4^2 & = & \langle x_1, y_2, h_1 + h_2\rangle, \\ L_5^1 & = & \langle x_1, x_2, x_3\rangle, \end{array} \tag{59}$$

for $\alpha \neq \pm 1$. The subscripts correspond to those of the appropriate isomorphism type in Equation (5). Further, $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ (resp. $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$) is equivalent to $L^{1,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$ (resp. $L^{2,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$) if and only if $\alpha = \beta$ or $\alpha\beta = 1$.

Proof. Let L be a solvable, 3-dimensional subalgebra of A_2. Then, by Lemma 6.4,

$$L = \langle z \rangle \Subset K, \tag{60}$$

where K is a 2-dimensional (solvable) subalgebra, and $z \in L$. By Theorems 6.2 and 6.3, after conjugation by an element of $SL(3, \mathbb{C})$, we have the following cases.

Case 1. $L = \langle z \rangle \Subset K_1^1 = \langle z \rangle \Subset \langle x_1 + x_2, x_3\rangle$. The conditions $[z, x_1 + x_2] \subseteq \langle x_1 + x_2, x_3\rangle$, and $[z, x_3] \subseteq \langle x_1 + x_2, x_3\rangle$ imply that $z = a(h_1 + h_2) + bx_1 + cx_2 + dx_3$, for some $a, b, c, d \in \mathbb{C}$, not all zero. The values of a, b, and c yield the following two possibilities:

$$L \sim \langle x_1 + x_2, x_3, h_1 + h_2\rangle, \quad \text{or } \langle x_1, x_2, x_3\rangle. \tag{61}$$

The subalgebra $\langle x_1 + x_2, x_3, h_1 + h_2\rangle$ is isomorphic to $L_{3,-\frac{2}{9}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + x_2 + x_3, \\ z_2 & \mapsto & \frac{1}{3}(x_1 + x_2) + \frac{2}{3}x_3, \\ z_3 & \mapsto & \frac{1}{3}(h_1 + h_2). \end{array} \tag{62}$$

The subalgebra $\langle x_1, x_2, x_3\rangle$ is isomorphic to L_5 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1, \\ z_2 & \mapsto & x_3, \\ z_3 & \mapsto & x_2. \end{array} \tag{63}$$

Define

$$\begin{array}{rcl} L^1_{3,-\frac{2}{9}} & \equiv & \langle x_1 + x_2, x_3, h_1 + h_2\rangle, \\ L^1_5 & \equiv & \langle x_1, x_2, x_3\rangle. \end{array} \tag{64}$$

Case 2. $L = \langle z\rangle \Subset K^2_1 = \langle z\rangle \Subset \langle x_1, h_1 + 2h_2\rangle$. The conditions $[z, x_1] \subseteq \langle x_1, h_1 + 2h_2\rangle$, and $[z, h_1 + 2h_2] \subseteq \langle x_1, h_1 + 2h_2\rangle$ imply that $z = ah_1 + bh_2 + cx_1$, for some $a, b, c \in \mathbb{C}$, not all zero. Then, $L = \langle x_1, h_1, h_2\rangle$. The subalgebra $\langle x_1, h_1, h_2\rangle$ is isomorphic to $L_{3,0}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & h_1 + 2h_2 + x_1, \\ z_2 & \mapsto & x_1, \\ z_3 & \mapsto & \frac{1}{2}h_1 + x_1. \end{array} \tag{65}$$

Define

$$L^1_{3,0} \equiv \langle x_1, h_1, h_2\rangle. \tag{66}$$

Case 3. $L = \langle z\rangle \Subset K^3_1 = \langle z\rangle \Subset \langle x_1, x_3\rangle$. The conditions $[z, x_1] \subseteq \langle x_1, x_3\rangle$, and $[z, x_3] \subseteq \langle x_1, x_3\rangle$ imply that $L = \langle x_1, x_3, ax_2 + by_2 + ch_1 + dh_2\rangle$, $a, b, c, d \in \mathbb{C}$, not all zero. In matrix form, we have

$$ax_2 + by_2 + ch_1 + dh_2 = \begin{pmatrix} c & 0 & 0 \\ 0 & -c+d & a \\ 0 & b & -d \end{pmatrix}. \tag{67}$$

We now consider the cases $4ab + c^2 - 4cd + 4d^2 \neq 0$, and $4ab + c^2 - 4cd + 4d^2 = 0$.

Case 3.1. $4ab + c^2 - 4cd + 4d^2 \neq 0$. In this case $ax_2 + by_2 + ch_1 + dh_2$ is diagonalizable, as is its lower right 2×2 block matrix, of course. For any block matrix

$$A = \begin{pmatrix} \frac{1}{\det(G)} & 0 \\ 0 & G \end{pmatrix} \in SL(3, \mathbb{C}), \tag{68}$$

where $G \in GL(2, \mathbb{C})$, $A^{-1}\langle x_1, x_3\rangle A = \langle x_1, x_3\rangle$. We may choose such a block matrix A which diagonalizes the lower right 2×2 block of $ax_2 + by_2 + ch_1 + dh_2$ such that $A^{-1}\langle ax_2 + by_2 + ch_1 + dh_2\rangle A = \langle(\alpha - 1)h_1 + \alpha h_2\rangle$, for some $\alpha \in \mathbb{C}$. We now proceed in cases based on the value of α.

<u>Case 3.1.1</u>. $\alpha = -1$. Then L is equivalent to $\langle x_1, x_3, 2h_1 + h_2 \rangle$, which is isomorphic to L_2 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1, \\ z_2 & \mapsto & x_3, \\ z_3 & \mapsto & \frac{1}{3}(2h_1 + h_2). \end{array} \tag{69}$$

Define

$$L_2^1 \equiv \langle x_1, x_3, 2h_1 + h_2 \rangle. \tag{70}$$

<u>Case 3.1.2</u>. $\alpha = 1$. Then L is equivalent to $\langle x_1, x_3, h_2 \rangle$, which is isomorphic to L_4 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + x_3, \\ z_2 & \mapsto & x_1 - x_3, \\ z_3 & \mapsto & -h_2. \end{array} \tag{71}$$

Define

$$L_4^1 \equiv \langle x_1, x_3, h_2 \rangle. \tag{72}$$

<u>Case 3.1.3</u>. $\alpha \neq \pm 1$. The subalgebra $\langle x_1, x_3, (\alpha-1)h_1 + \alpha h_2 \rangle$ is isomorphic to $L_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + x_3, \\ z_2 & \mapsto & \frac{\alpha-2}{3(\alpha-1)}x_1 + \frac{2\alpha-1}{3(\alpha-1)}x_3, \\ z_3 & \mapsto & \frac{1}{3(\alpha-1)}((\alpha-1)h_1 + \alpha h_2). \end{array} \tag{73}$$

For $\alpha \neq \pm 1$, define

$$L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} \equiv \langle x_1, x_3, (\alpha-1)h_1 + \alpha h_2 \rangle. \tag{74}$$

We now establish that $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is equivalent to $L^{1,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$ if and only if $\alpha = \beta$ or $\alpha\beta = 1$. First suppose that $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is equivalent to $L^{1,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$, then these subalgebras are isomorphic so that $-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2} = -\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}$. Then

$$(2\alpha-1)(\alpha-2)(\beta-1)^2 = (2\beta-1)(\beta-2)(\alpha-1)^2. \tag{75}$$

Expanding, and simplifying Eq. (75) yields

$$\begin{array}{rl} & \alpha^2\beta - \alpha\beta^2 - \alpha + \beta = 0 \\ \Rightarrow & (\alpha\beta - 1)(\alpha - \beta) = 0 \\ \Rightarrow & \alpha\beta = 1 \text{ or } \alpha = \beta. \end{array} \tag{76}$$

Now suppose that $\alpha\beta = 1$, or $\alpha = \beta$, then $-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2} = -\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}$ so that $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is isomorphic to $L^{1,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$. Further, they are equivalent via conjugation by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in SL(3,\mathbb{C}), \tag{77}$$

in the case that $\alpha\beta = 1$.

We now establish that $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is not equivalent to any subalgebra defined previously in this proof. In particular, we must consider $\alpha \neq \pm 1$ for which $-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2} = -\frac{2}{9}$, or 0. That is, we must consider $\alpha = 0, \frac{1}{2}$, and 2. But, we may exclude the $\alpha = \frac{1}{2}$ case since $L^{1,\frac{1}{2}}_{3,0}$ is equivalent to $L^{1,2}_{3,0}$.

If $\alpha = 0$, then $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = L^{1,0}_{3,-\frac{2}{9}}$. Straightforward calculation shows that $L^{1,0}_{3,-\frac{2}{9}}$ is inequivalent to $L^{1}_{3,-\frac{2}{9}}$.

If $\alpha = 2$, then $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = L^{1,2}_{3,0}$. Straightforward calculation shows that $L^{1,2}_{3,0}$ is inequivalent to $L^{1}_{3,0}$.

Case 3.2. $4ab + c^2 - 4cd + 4d^2 = 0$. Then $d = \frac{1}{2}c \pm \sqrt{-ab}$, so that

$$L = \left\langle x_1, x_3, ax_2 + by_2 + ch_1 + \left(\frac{1}{2}c \pm \sqrt{-ab}\right) h_2 \right\rangle. \tag{78}$$

Case 3.2.1. $a \neq 0$, $b \neq 0$, $c \neq 0$. Then, conjugation by

$$A = \frac{1}{\sqrt[3]{-b}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm\sqrt{-ab} & 1 \\ 0 & b & 0 \end{pmatrix} \in SL(3,\mathbb{C}), \tag{79}$$

yields $L \sim \langle x_1, x_3, ch_1 + \frac{c}{2}h_2 + x_2 \rangle$. After an additional appropriate conjugation we get $L \sim \langle x_1, x_3, 2h_1 + h_2 + x_2 \rangle$, which is isomorphic to $L_{3,-\frac{1}{4}}$ via the isomorphism given by

$$\begin{aligned} z_1 &\mapsto x_1 + x_3, \\ z_2 &\mapsto \tfrac{1}{2}x_1 + \tfrac{2}{3}x_3, \\ z_3 &\mapsto \tfrac{1}{6}(2h_1 + h_2 + x_2). \end{aligned} \tag{80}$$

Define

$$L^{1}_{3,-\frac{1}{4}} \equiv \langle x_1, x_3, 2h_1 + h_2 + x_2 \rangle. \tag{81}$$

Note that $L_{3,-\frac{1}{4}}$ is not isomorphic–let alone equivalent–to any subalgebra defined previously in this proof. In particular, $L_{3,-\frac{1}{4}} \not\cong L_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ for all $\alpha \neq \pm 1$.

<u>Case 3.2.2</u>. At least one of $a = 0$, $b = 0$, or $c = 0$. If precisely two of a, b, c are zero, then L is equal to $\langle x_1, x_3, 2h_1 + h_2\rangle = L_2^1$, $\langle x_1, x_2, x_3\rangle = L_5^1$, or $\langle x_1, x_3, y_2\rangle \sim L_5^1$.

If $b = 0$, and $a, c \neq 0$; or if $a = 0$, and $b, c \neq 0$, then, after appropriate conjugation in $SL(3,\mathbb{C})$, $L \sim \langle x_1, x_3, 2h_1 + h_2 + x_2\rangle = L^1_{3,-\frac{1}{4}}$.

If $c = 0$, and $a, b \neq 0$, then, after appropriate conjugation in $SL(3,\mathbb{C})$, L is equivalent to $\langle x_1, y_2, y_3\rangle$, which is equivalent to L_5^1.

<u>Case 4</u>. $L = \langle z\rangle \Subset K_1^4 = \langle z\rangle \Subset \langle x_1, y_2\rangle$. The conditions $[z, x_1] \subseteq \langle x_1, y_2\rangle$, and $[z, y_2] \subseteq \langle x_1, y_2\rangle$ imply that $L = \langle x_1, y_2, ax_3 + by_3 + ch_1 + dh_2\rangle$, $a, b, c, d \in \mathbb{C}$, not all zero. In matrix form, we have

$$ax_3 + by_3 + ch_1 + dh_2 = \begin{pmatrix} c & 0 & -a \\ 0 & -c+d & 0 \\ -b & 0 & -d \end{pmatrix}. \tag{82}$$

We now consider the cases $4ab+c^2+2cd+d^2 \neq 0$, and $4ab+c^2+2cd+d^2 = 0$.

<u>Case 4.1</u>. $4ab + c^2 + 2cd + d^2 \neq 0$. In this case $ax_3 + by_3 + ch_1 + dh_2$ is diagonalizable, as is the 2×2 matrix comprised of its four corners. For any matrix

$$A = \begin{pmatrix} g_{11} & 0 & g_{12} \\ 0 & \frac{1}{\det(G)} & 0 \\ g_{21} & 0 & g_{22} \end{pmatrix} \in SL(3,\mathbb{C}), \tag{83}$$

where $G = [g_{ij}] \in GL(2,\mathbb{C})$, $A^{-1}\langle x_1, y_2\rangle A = \langle x_1, y_2\rangle$. We may choose a matrix A of the form above in Eq. (83) which diagonalizes the 2×2 matrix comprised of the four corners of $ax_3 + by_3 + ch_1 + dh_2$ such that $A^{-1}\langle ax_3 + by_3 + ch_1 + dh_2\rangle A = \langle h_1 + \alpha h_2\rangle$, for some $\alpha \in \mathbb{C}$. We now proceed in cases based on the value of α.

<u>Case 4.1.1</u>. $\alpha = -1$. In this case $L = \langle x_1, y_2, h_1 - h_2\rangle$, which is isomorphic to L_2 via the isomorphism given by

$$\begin{aligned} z_1 &\mapsto x_1 + y_2, \\ z_2 &\mapsto x_1 - y_2, \\ z_3 &\mapsto \tfrac{1}{3}(h_1 - h_2). \end{aligned} \tag{84}$$

Define

$$L_2^2 \equiv \langle x_1, y_2, h_1 - h_2\rangle. \tag{85}$$

Straightforward computation shows that L_2^1, and L_2^2 are inequivalent.

<u>Case 4.1.2</u>. $\alpha = 1$. In this case $L = \langle x_1, y_2, h_1+h_2\rangle$, which is isomorphic to L_4 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + y_2, \\ z_2 & \mapsto & x_1 - y_2, \\ z_3 & \mapsto & h_1 + h_2. \end{array} \tag{86}$$

Define

$$L_4^2 \equiv \langle x_1, y_2, h_1 + h_2\rangle. \tag{87}$$

Straightforward computation shows that L_4^1 and L_4^2 are inequivalent.
<u>Case 4.1.3</u>. $\alpha \neq \pm 1$. In this case the subalgebra $L = \langle x_1, y_2, h_1 + \alpha h_2\rangle$ is isomorphic to $L_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + y_2, \\ z_2 & \mapsto & \frac{1}{3(\alpha-1)}((\alpha-2)x_1 + (2\alpha-1)y_2), \\ z_3 & \mapsto & -\frac{1}{3(\alpha-1)}(h_1 + \alpha h_2). \end{array} \tag{88}$$

For $\alpha \neq \pm 1$, define

$$L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} \equiv \langle x_1, y_2, h_1 + \alpha h_2\rangle. \tag{89}$$

For $\alpha, \beta \neq \pm 1$, we establish that $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is equivalent to $L^{2,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$ if and only if $\alpha = \beta$, or $\alpha\beta = 1$, as we did in Case 3.1.3.

We now establish that $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ is not equivalent to any subalgebra defined previously in this proof. First, we must consider $\alpha \neq \pm 1$ for which $-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2} = -\frac{1}{4}, -\frac{2}{9}$, or 0. That is, we must consider $\alpha = 0, 2$; as in Case 3.3.1 we may exclude the $\alpha = \frac{1}{2}$ case.

Second, we must consider $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$, and $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$ for $\alpha \neq \pm 1$. But, straightforward computation shows these two subalgebras to be inequivalent.

If $\alpha = 0$, then $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = L^{2,0}_{3,-\frac{2}{9}}$. Straightforward calculation shows that $L^{2,0}_{3,-\frac{2}{9}}$ is inequivalent to $L^1_{3,-\frac{2}{9}}$, and $L^{1,0}_{3,-\frac{2}{9}}$.

If $\alpha = 2$, then $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = L^{2,2}_{3,0}$. Straightforward calculation shows that $L^{2,2}_{3,0}$ is inequivalent to $L^1_{3,0}$, and $L^{1,2}_{3,0}$.
<u>Case 4.2</u>. $4ab + c^2 + 2cd + d^2 = 0$. Then $d = -c \pm 2\sqrt{-ab}$, so that

$$L = \langle x_1, y_2, ax_3 + by_3 + ch_1 + (-c \pm 2\sqrt{-ab})h_2\rangle. \tag{90}$$

Case 4.2.1. $b \neq 0$. Then conjugation by

$$A = \frac{1}{\sqrt[3]{-b}} \begin{pmatrix} 0 & \pm\sqrt{-ab} & 1 \\ 1 & 0 & 0 \\ 0 & -b & 0 \end{pmatrix} \in SL(3, \mathbb{C}), \tag{91}$$

yields $\langle y_1, y_3, (-2c \pm 2\sqrt{-ab})h_1 + (c \mp \sqrt{-ab})h_2 + x_2 \rangle$. After an additional appropriate conjugation we get $L \sim \langle y_1, y_3, 2h_1 + h_2 + x_2 \rangle$, or $\langle x_2, y_1, y_3 \rangle$.

The subalgebra $\langle y_1, y_3, 2h_1 + h_2 + x_2 \rangle$ is isomorphic to $L_{3,-\frac{1}{4}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & y_1 + y_3, \\ z_2 & \mapsto & \frac{2}{3}y_1 + \frac{1}{2}y_3, \\ z_3 & \mapsto & -\frac{1}{6}(2h_1 + h_2 + x_2). \end{array} \tag{92}$$

Define

$$L^2_{3,-\frac{1}{4}} \equiv \langle y_1, y_3, 2h_1 + h_2 + x_2 \rangle. \tag{93}$$

Direct computation shows that $L^1_{3,-\frac{1}{4}}$, and $L^2_{3,-\frac{1}{4}}$ are inequivalent.

The subalgebra $\langle x_2, y_1, y_3 \rangle$ is isomorphic to L_5 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_2, \\ z_2 & \mapsto & y_1, \\ z_3 & \mapsto & y_3. \end{array} \tag{94}$$

The subalgebra $\langle x_2, y_1, y_3 \rangle$ is equivalent to L^1_5 via conjugation by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \in SL(3, \mathbb{C}). \tag{95}$$

Case 4.2.2. $b = 0$. If $a = 0$, then $L = \langle x_1, y_2, h_1 - h_2 \rangle = L^2_2$. If $c = 0$, then $L = \langle x_1, x_3, y_2 \rangle$, which is isomorphic to L_5 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & y_2, \\ z_2 & \mapsto & -x_1, \\ z_3 & \mapsto & x_3. \end{array} \tag{96}$$

The subalgebra $\langle x_1, x_3, y_2 \rangle$ is equivalent to L^1_5.

If $a, c \neq 0$, then, after appropriate conjugation, $L \sim \langle x_1, y_2, h_1 - h_2 + x_3 \rangle$, which is isomorphic to $L_{3,-\frac{1}{4}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & x_1 + y_2, \\ z_2 & \mapsto & \frac{1}{3}x_1 + \frac{1}{2}y_2, \\ z_3 & \mapsto & \frac{1}{6}(h_1 - h_2 + x_3). \end{array} \tag{97}$$

The subalgebra $\langle x_1, y_2, h_1 - h_2 + x_3\rangle$ is equivalent to $L^2_{3,-\frac{1}{4}}$.

Case 5. $L = \langle z\rangle \Cap K_1^5 = \langle z\rangle \Cap \langle h_1, h_2\rangle$. The conditions $[z, h_1] \subseteq \langle h_1, h_2\rangle$, and $[z, h_2] \subseteq \langle h_1, h_2\rangle$ imply that z is linear combination of h_1 and h_2, which is a contradiction to the fact that L is 3-dimensional. Hence, this case does not yield a subalgebra.

Case 6. $L = \langle z\rangle \Cap K_2^1 = \langle z\rangle \Cap \langle x_1 + x_2, h_1 + h_2\rangle$. The conditions $[z, x_1 + x_2] \subseteq \langle x_1 + x_2, h_1 + h_2\rangle$, and $[z, h_1 + h_2] \subseteq \langle x_1 + x_2, h_1 + h_2\rangle$ imply that z is a linear combination of $x_1 + x_2$, and $h_1 + h_2$, which is a contradiction to the fact that L is 3-dimensional. Hence, this case does not yield a subalgebra.

Case 7. $L = \langle z\rangle \Cap K_2^2 = \langle z\rangle \Cap \langle x_1, -\frac{1}{3}h_1 + \frac{1}{3}h_2 + x_3\rangle$. The conditions $[z, x_1 + x_2] \subseteq \langle x_1 + x_2, h_1 + h_2\rangle$, and $[z, h_1 + h_2] \subseteq \langle x_1 + x_2, h_1 + h_2\rangle$ imply that z is a linear combination of $-\frac{1}{3}h_1 + \frac{1}{3}h_2$, and x_3. Hence, $L = \langle x_1, x_3, h_1 - h_2\rangle$, which is isomorphic to $L_{3,0}$ via the isomorphism given by

$$\begin{array}{lcl} z_1 & \mapsto & x_1 + x_3, \\ z_2 & \mapsto & x_1, \\ z_3 & \mapsto & \frac{1}{3}(h_1 - h_2). \end{array} \tag{98}$$

$L^{1,2}_{3,0}$ is equivalent to $\langle x_1, x_3, h_1 - h_2\rangle$ via conjugation by

$$\begin{pmatrix} 1 & 1 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{C}). \tag{99}$$

Case 8. $L = \langle z\rangle \Cap K_2^3 = \langle z\rangle \Cap \langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2\rangle$. The conditions $[z, x_1] \subseteq \langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2\rangle$, and $[z, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2] \subseteq \langle x_1, -\frac{2}{3}h_1 - \frac{1}{3}h_2 + y_2\rangle$ imply that $z = a(2h_1 + h_2) + bx_1 + cy_2$, for $a, b, c \in \mathbb{C}$. The requirement that $\dim L = 3$ then implies $L = \langle x_1, y_2, 2h_1 + h_2\rangle$, which is isomorphic to $L_{3,0}$ via the isomorphism given by

$$\begin{array}{lcl} z_1 & \mapsto & x_1 + y_2, \\ z_2 & \mapsto & x_1, \\ z_3 & \mapsto & \frac{1}{3}(2h_1 + h_2). \end{array} \tag{100}$$

We have $\langle x_1, y_2, 2h_1 + h_2\rangle = L^{2,\frac{1}{2}}_{3,0}$.

Case 9. $L = \langle z\rangle \Cap K_2^{4,\alpha} = \langle z\rangle \Cap \langle x_1, \alpha h_1 + (2\alpha + 1)h_2\rangle$. The conditions $[z, x_1] \subseteq \langle x_1, \alpha h_1 + (2\alpha + 1)h_2\rangle$, and $[z, \alpha h_1 + (2\alpha + 1)h_2] \subseteq \langle x_1, \alpha h_1 + (2\alpha + 1)h_2\rangle$ imply that $L = \langle x_1, \alpha h_1 + (2\alpha + 1)h_2, ax_3 + by_2 + ch_1 + dh_2\rangle$, $a, b, c, d \in \mathbb{C}$, such that $a(3\alpha + 1) = 0$, and $b(3\alpha + 2) = 0$.

The value of α ($\alpha = -\frac{1}{3}$; $\alpha = -\frac{2}{3}$; or $\alpha \neq -\frac{1}{3}, -\frac{2}{3}$) yields the following equivalence possibilities for L:

$$\begin{array}{lcl} \langle x_1, x_3, h_1 - h_2 \rangle & \sim & L^{1,2}_{3,0}, \\ \langle x_1, y_2, 2h_1 + h_2 \rangle & = & L^{2,\frac{1}{2}}_{3,0}, \\ \langle x_1, h_1, h_2 \rangle & = & L^{1}_{3,0}. \end{array} \tag{101}$$

□

Remark 6.6. The (complexification) of the Euclidean algebra $\mathfrak{e}(2) \cong \mathfrak{so}(2) \Subset \mathbb{R}^2$ is isomorphic to L_4. Hence, by Theorem 6.5, there are precisely two nonequivalent copies of (the complexification of) $\mathfrak{e}(2)$ in A_2. This point is also implied by the results of the authors in [3].

6.5. **Four-dimensional, solvable subalgebras of A_2.**

Theorem 6.7. *A classification of the 4-dimensional solvable subalgebras of A_2, up to inner automorphism, is given by:*

$$\begin{array}{lcl} M^1_8 & = & \langle x_1, x_3, h_1, h_2 \rangle, \\ M^2_8 & = & \langle x_1, y_2, h_1, h_2 \rangle, \\ M^1_{12} & = & \langle x_1, x_2, x_3, h_1 + h_2 \rangle, \\ M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}} & = & \langle x_1, x_2, x_3, \alpha h_1 + h_2 \rangle, \\ M^2_{13,0} & = & \langle x_2, y_1, y_3, 2h_1 + h_2 \rangle, \\ M^2_{13,2} & = & \langle x_1, x_2, x_3, h_1 \rangle, \\ M^1_{14} & = & \langle x_1, x_2, x_3, h_1 - h_2 \rangle, \end{array} \tag{102}$$

where $\alpha \neq \pm 1$; and $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ and $M^{1,\beta}_{13,\frac{(2\beta-1)(\beta-2)}{(\beta+1)^2}}$ are equivalent if and only if $\alpha = \beta$. The subscripts correspond to those of the appropriate isomorphism type in Equation (6).

Proof. Let M be a solvable 4-dimensional subalgebra of A_2. Then, by Lemma 6.4,

$$M = \langle z \rangle \Subset L, \tag{103}$$

where L is a 3-dimensional solvable subalgebra, and $z \in M$. By Theorem 6.5, we have the following cases.

<u>Case 1</u>. $M = \langle z \rangle \Subset L^1_{3,-\frac{2}{9}} = \langle z \rangle \Subset \langle x_1 + x_2, x_3, h_1 + h_2 \rangle$. Then, z is a linear combination of $x_1 + x_2$, x_3, and $h_1 + h_2$, which is a contradiction to the dimension of M. Hence, this case does not yield a subalgebra.

<u>Case 2</u>. $M = \langle z \rangle \Subset L^1_{3,0} = \langle z \rangle \Subset \langle x_1, h_1, h_2 \rangle$. Then, z is a linear combination of x_1, h_1, and h_2, which is a contradiction to the dimension of M. Hence, this case does not yield a subalgebra.

Case 3. $M = \langle z \rangle \Subset L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = \langle z \rangle \Subset \langle x_1, x_3, (\alpha-1)h_1 + \alpha h_2 \rangle$, $\alpha \neq \pm 1$. Then, $M = \langle x_1, x_3, h_1, h_2 \rangle$, which is isomorphic to M_8 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & \frac{1}{3}(h_1 - h_2), \\ z_2 & \mapsto & x_1, \\ z_3 & \mapsto & \frac{1}{3}(h_1 + 2h_2), \\ z_4 & \mapsto & x_3. \end{array} \tag{104}$$

Define

$$M_8^1 \equiv \langle x_1, x_3, h_1, h_2 \rangle. \tag{105}$$

Case 4. $M = \langle z \rangle \Subset L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} = \langle z \rangle \Subset \langle x_1, y_2, h_1 + \alpha h_2 \rangle$, $\alpha \neq \pm 1$. Then, $M = \langle x_1, y_2, h_1, h_2 \rangle$, which is isomorphic to M_8 via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & -\frac{1}{3}(h_1 + 2h_2), \\ z_2 & \mapsto & y_2, \\ z_3 & \mapsto & \frac{1}{3}(2h_1 + h_2), \\ z_4 & \mapsto & x_1. \end{array} \tag{106}$$

Define

$$M_8^2 \equiv \langle x_1, y_2, h_1, h_2 \rangle. \tag{107}$$

Straightforward computation establishes that M_8^1 and M_8^2 are inequivalent.

Case 5. $M = \langle z \rangle \Subset L_4^1 = \langle z \rangle \Subset \langle x_1, x_3, h_2 \rangle$. Then, $M = \langle x_1, x_3, h_1, h_2 \rangle = M_8^1$.

Case 6. $M = \langle z \rangle \Subset L_4^2 = \langle z \rangle \Subset \langle x_1, y_2, h_1 + h_2 \rangle$. Then, $M = \langle x_1, y_2, h_1, h_2 \rangle = M_8^2$.

Case 7. $M = \langle z \rangle \Subset L_4^2 = \langle z \rangle \Subset \langle x_1, x_2, x_3 \rangle$. Then, $M = \langle x_1, x_2, x_3, \alpha h_1 + h_2 \rangle$ or $\langle x_1, x_2, x_3, h_1 + \alpha h_2 \rangle$, for some $\alpha \in \mathbb{C}$. Up to equivalence, this reduces to $\langle x_1, x_2, x_3, \alpha h_1 + h_2 \rangle$ for $\alpha \in \mathbb{C}$ or $\langle x_1, x_2, x_3, h_1 \rangle$.

For $\alpha \neq \pm 1$, $\langle x_1, x_2, x_3, \alpha h_1 + h_2 \rangle$ is isomorphic to $M_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ via the isomorphism given by

$$\begin{array}{rcl} z_1 & \mapsto & \frac{2\alpha-1}{\alpha+1}x_1 - \frac{\alpha-2}{\alpha+1}x_2, \\ z_2 & \mapsto & \frac{3(\alpha-1)}{\alpha+1}x_3, \\ z_3 & \mapsto & x_1 + x_2, \\ z_4 & \mapsto & \frac{1}{\alpha+1}(\alpha h_1 + h_2). \end{array} \tag{108}$$

For $\alpha \neq \pm 1$, define

$$M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}} \equiv \langle x_1, x_2, x_3, \alpha h_1 + h_2 \rangle. \tag{109}$$

Straightforward computation establishes that $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ and $M^{1,\beta}_{13,\frac{(2\beta-1)(\beta-2)}{(\beta+1)^2}}$ are equivalent if and only if $\alpha = \beta$.

The subalgebra $\langle x_1, x_2, x_3, h_1\rangle$ is isomorphic to $M_{13,2}$ via the isomorphism given by

$$\begin{array}{lll} z_1 & \mapsto & 2x_1 - x_2, \\ z_2 & \mapsto & 3x_3, \\ z_3 & \mapsto & x_1 + x_2, \\ z_4 & \mapsto & h_1. \end{array} \tag{110}$$

Define

$$M^2_{13,2} \equiv \langle x_1, x_2, x_3, h_1\rangle. \tag{111}$$

Then $M^2_{13,2}$ and $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ are isomorphic precisely when $\alpha = 0$. Straightforward computation shows that $M^2_{13,2}$ and $M^{1,0}_{13,2}$ are inequivalent.

The subalgebra $\langle x_1, x_2, x_3, h_1 + h_2\rangle$ is isomorphic to M_{12} via the isomorphism given by

$$\begin{array}{lll} z_1 & \mapsto & 2x_1 + x_2, \\ z_2 & \mapsto & x_3, \\ z_3 & \mapsto & x_1 + x_2, \\ z_4 & \mapsto & h_1 + h_2. \end{array} \tag{112}$$

Define

$$M^1_{12} \equiv \langle x_1, x_2, x_3, h_1 + h_2\rangle. \tag{113}$$

The subalgebra $\langle x_1, x_2, x_3, h_1 - h_2\rangle$ is isomorphic to M_{14} via the isomorphism given by

$$\begin{array}{lll} z_1 & \mapsto & x_1 - x_2, \\ z_2 & \mapsto & 2x_3, \\ z_3 & \mapsto & x_1 + x_2, \\ z_4 & \mapsto & \frac{1}{3}(h_1 - h_2). \end{array} \tag{114}$$

Define

$$M^1_{14} \equiv \langle x_1, x_2, x_3, h_1 - h_2\rangle. \tag{115}$$

Case 8. $M = \langle z\rangle \Subset L^1_{3,-\frac{1}{4}} = \langle z\rangle \Subset \langle x_1, x_3, 2h_1 + h_2 + x_2\rangle$. Then, $M = \langle x_1, x_2, x_3, 2h_1 + h_2\rangle = M^{1,2}_{13,0}$.

Case 9. $M = \langle z\rangle \Subset L^2_{3,-\frac{1}{4}} = \langle z\rangle \Subset \langle y_1, y_3, 2h_1 + h_2 + x_2\rangle$. Then, $M = \langle x_2, y_1, y_3, 2h_1 + h_2\rangle$, which is isomorphic to $M_{13,0}$ via the isomorphism

given by

$$\begin{array}{lll} z_1 & \mapsto & y_3, \\ z_2 & \mapsto & -y_1, \\ z_3 & \mapsto & x_2 + y_3, \\ z_4 & \mapsto & -\frac{1}{3}(2h_1 + h_2). \end{array} \tag{116}$$

Define

$$M^2_{13,0} \equiv \langle x_2, y_1, y_3, 2h_1 + h_2 \rangle. \tag{117}$$

Straightforward computation establishes that $M^2_{13,0}$, and $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$ are inequivalent for all $\alpha \neq \pm 1$.

Case 10. $M = \langle z \rangle \Subset L^1_2 = \langle z \rangle \Subset \langle x_1, x_3, 2h_1 + h_2 \rangle$. Then, $M = \langle x_1, x_3, 2h_1 + h_2, ax_2 + by_2 + ch_1 + dh_2 \rangle$, for $a, b, c, d \in \mathbb{C}$, not all zero. In matrix form $ax_2 + by_2 + ch_1 + dh_2$ is equal to the block matrix

$$ax_2 + by_2 + ch_1 + dh_2 = \left(\begin{array}{ccc} c & 0 & 0 \\ 0 & -c + d & a \\ 0 & b & -d \end{array} \right). \tag{118}$$

Case 10.1. $4ab + c^2 - 4cd + 4d^2 \neq 0$. The eigenvalues of the lower right 2×2 block of $ax_2 + by_2 + ch_1 + dh_2$ are

$$\begin{array}{l} -\frac{1}{2}c + \frac{1}{2}\sqrt{4ab + c^2 - 4cd + 4d^2}, \text{ and} \\ -\frac{1}{2}c - \frac{1}{2}\sqrt{4ab + c^2 - 4cd + 4d^2}, \end{array} \tag{119}$$

which are distinct for $4ab+c^2-4cd+4d^2 \neq 0$. Hence, $ax_2+by_2+ch_1+dh_2$ is diagonalizable. Note that the lower right 2×2 blocks of $2h_1 + h_2$ and $ax_2 + by_2 + ch_1 + dh_2$ commute. Hence, the lower right 2×2 blocks of these elements are simultaneously diagonalizable, via conjugation by some $G \in GL(2, \mathbb{C})$. If we set

$$A = \left(\begin{array}{cc} \frac{1}{\det(G)} & 0 \\ 0 & G \end{array} \right) \in SL(3, \mathbb{C}), \tag{120}$$

then $A^{-1}\langle x_1, x_3 \rangle A = \langle x_1, x_3 \rangle$, so that $A^{-1}\langle x_1, x_3, 2h_1 + h_2, ax_2 + by_2 + ch_1 + dh_2 \rangle A = \langle x_1, x_3, h_1, h_2 \rangle = M^1_8$.

Case 10.2. $4ab + c^2 - 4cd + 4d^2 = 0$. Then $d = \frac{1}{2}c \pm \sqrt{-ab}$.

Case 10.2.1. $b \neq 0$. Let

$$G = \left(\begin{array}{cc} \pm\sqrt{-ab} & 1 \\ b & 0 \end{array} \right) \in GL(2, \mathbb{C}), \tag{121}$$

where the sign in G varies with the sign in d. Then conjugation by

$$A = \left(\begin{array}{cc} \frac{1}{\det(G)} & 0 \\ 0 & G \end{array} \right) \in SL(3, \mathbb{C}), \tag{122}$$

yields $\langle x_1, x_3, 2h_1+h_2, ch_1+\frac{1}{2}ch_2+x_2\rangle = \langle x_1, x_2, x_3, 2h_1+h_2\rangle = M^{1,2}_{13,0}$.
Case 10.2.2. $b=0$. Then $M = \langle x_1, x_3, 2h_1+h_2, ch_1+\frac{1}{2}ch_2+ax_2\rangle$, and, by dimension considerations, $a \neq 0$. Hence, $M = \langle x_1, x_2, x_3, 2h_1+h_2\rangle$ $= M^{1,2}_{13,0}$.
Case 11. $M = \langle z\rangle \Subset L^2_2 = \langle z\rangle \Subset \langle x_1, y_2, h_1-h_2\rangle$. Then, $M = \langle x_1, y_2, h_1-h_2, ax_3+by_3+ch_1+dh_2\rangle$, for $a, b, c, d \in \mathbb{C}$, not all zero. In matrix form, $ax_3+by_3+ch_1+dh_2$ is equal to

$$ax_3+by_3+ch_1+dh_2 = \left(\begin{array}{ccc} c & 0 & -a \\ 0 & -c+d & 0 \\ -b & 0 & -d \end{array}\right). \tag{123}$$

Case 11.1. $4ab+c^2+2cd+d^2 \neq 0$. The eigenvalues of the 2×2 matrix formed from the outer corners of $ax_3+by_3+ch_1+dh_2$ are

$$\begin{array}{l} \frac{1}{2}c-\frac{1}{2}d+\frac{1}{2}\sqrt{4ab+c^2+2cd+d^2}, \text{ and} \\ \frac{1}{2}c-\frac{1}{2}d-\frac{1}{2}\sqrt{4ab+c^2+2cd+d^2}, \end{array} \tag{124}$$

which are distinct for $4ab+c^2+2cd+d^2 \neq 0$. Hence, $ax_3+by_3+ch_1+dh_2$ is diagonalizable. Note that the 2×2 matrices formed from the corners of h_1-h_2 and $ax_3+by_3+ch_1+dh_2$ commute. Hence, the 2×2 matrices formed from the corners of these elements are simultaneously diagonalizable, via conjugation by some $G=[g_{ij}] \in GL(2,\mathbb{C})$. If we set

$$A = \left(\begin{array}{ccc} g_{11} & 0 & g_{12} \\ 0 & \frac{1}{\det(G)} & 0 \\ g_{21} & 0 & g_{22} \end{array}\right) \in SL(3,\mathbb{C}), \tag{125}$$

then $A^{-1}\langle x_1, y_2\rangle A = \langle x_1, y_2\rangle$, so that $A^{-1}\langle x_1, y_2, h_1-h_2, ax_3+by_3+ch_1+dh_2\rangle A {=} \langle x_1, y_2, h_1, h_2\rangle = M^2_8$.
Case 11.2. $4ab+c^2+2cd+d^2 = 0$. Then $d = -c \pm 2\sqrt{-ab}$.
Case 11.2.1. $b \neq 0$. Let

$$G = [g_{ij}] = \left(\begin{array}{cc} \pm\sqrt{-ab} & 1 \\ -b & 0 \end{array}\right) \in GL(2,\mathbb{C}), \tag{126}$$

where the sign in G varies with the sign in d. Then conjugation by

$$A = \left(\begin{array}{ccc} g_{11} & 0 & g_{12} \\ 0 & \frac{1}{\det(G)} & 0 \\ g_{21} & 0 & g_{22} \end{array}\right) \in SL(3,\mathbb{C}), \tag{127}$$

yields $\langle x_1, y_2, h_1-h_2, (c\mp\sqrt{-ab})h_1-(c\mp\sqrt{-ab})h_2+x_3\rangle = \langle x_1, x_3, y_2, h_1-h_2\rangle$, which is equivalent to $M^{1,\frac{1}{2}}_{13,0}$.

Case 11.2.2. $b = 0$. Then $M = \langle x_1, y_2, h_1 - h_2, ch_1 - ch_2 + ax_3 \rangle$, and, by dimension considerations, $a \neq 0$. Hence, $M = \langle x_1, x_3, y_2, h_1 - h_2 \rangle$, which is equivalent to $M_{13,0}^{1,\frac{1}{2}}$. □

6.6. **Five-dimensional, solvable subalgebras of A_2.** Since simple Lie algebras have unique, up to inner automorphism, maximal solvable subalgebras, Borel subalgebras, we have the following theorem.

Theorem 6.8. *The unique 5-dimensional solvable subalgebra of A_2, up to inner automorphism, is a Borel subalgebra:*

$$B \equiv \langle x_1, x_2, x_3, h_1, h_2 \rangle. \tag{128}$$

7. Conclusions

A classification of the semisimple subalgebras of A_2, the Lie algebra of traceless 3×3 matrices with complex entries, is well-known. In this article, we classified the solvable and Levi decomposable subalgebras of A_2, up to inner automorphism. By Levi's Theorem, this completes the classification of the subalgebras of A_2. The classification is summarized in Table 1.

Table 1. Classification of subalgebras of A_2, up to inner automorphism.

Dimension	Semisimple	Levi decomposable	Solvable
1	None	None	J^1, J^2, J^3, $J^{4,\alpha}$ where $J^{4,\alpha} \sim J^{4,\beta}$ if and only if $\alpha = 1-\beta$, $\alpha = \beta$, $\alpha = (-1+\alpha)\beta$, $\alpha = -(-1+\alpha)(-1+\beta)$, $\alpha\beta = 1$, or $-\alpha(-1+\beta) = 1$.
2	None	None	K_1^i, $1 \leq i \leq 5$, K_2^1, K_2^2, K_2^3, $K_2^{4,\alpha}$ where $K_2^{4,\alpha} \sim K_2^{4,\beta}$ if and only if $\alpha = \beta$.
3	A_1^1, A_1^2	None	L_2^1, L_2^2, $L^1_{3,-\frac{1}{4}}$, $L^2_{3,-\frac{1}{4}}$, $L^1_{3,-\frac{2}{9}}$, $L^1_{3,0}$, L_4^1, L_4^2, L_5^1, $L^{1,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$, $L^{2,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}}$, $\alpha \neq \pm 1$, where, for $i = 1, 2$, $L^{i,\alpha}_{3,-\frac{(2\alpha-1)(\alpha-2)}{9(\alpha-1)^2}} \sim L^{i,\beta}_{3,-\frac{(2\beta-1)(\beta-2)}{9(\beta-1)^2}}$ if and only if $\alpha = \beta$ or $\alpha\beta = 1$.
4	None	$(A_1 \oplus J)^1$	M_8^1, M_8^1, M_{12}^1, $M_{13,0}^2$, $M_{13,2}^2$, M_{14}^1, $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}}$, $\alpha \neq \pm 1$, where $M^{1,\alpha}_{13,\frac{(2\alpha-1)(\alpha-2)}{(\alpha+1)^2}} \sim M^{1,\beta}_{13,\frac{(2\beta-1)(\beta-2)}{(\beta+1)^2}}$ if and only if $\alpha = \beta$.
5	None	$(A_1 \inplus_{\varphi_1} K_1)^1$, $(A_1 \inplus_{\varphi_1} K_1)^2$	B
6	None	$(A_1 \inplus_{\varphi_2} L_2)^1$, $(A_1 \inplus_{\varphi_2} L_2)^2$	None

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